

MAS212 Linear Algebra I 2007-2008

Coursework 4 — Solutions

Exercise* 1. Recall that if $A = (a_{ij})_{2 \times 2}$, then

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}.$$

(a) If B is obtained from A by swapping row 1 and row 2, then

$$B = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix},$$

$$\text{so } \det(B) = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{21}a_{12}) = -\det(A).$$

(b) If B is obtained from A by multiplying row 1 by α , then

$$B = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$\text{so } \det(B) = \alpha a_{11}a_{22} - \alpha a_{21}a_{12} = \alpha(a_{11}a_{22} - a_{21}a_{12}) = \alpha \det(A).$$

(c) If B is obtained from A by adding r times row 2 to row 1, then

$$B = \begin{pmatrix} a_{11} + ra_{21} & a_{12} + ra_{22} \\ a_{21} & a_{22} \end{pmatrix},$$

$$\text{so } \det(B) = (a_{11} + ra_{21})a_{22} - a_{21}(a_{12} + ra_{22}) = a_{11}a_{22} + ra_{21}a_{22} - a_{21}a_{12} - ra_{21}a_{22} = a_{11}a_{22} - a_{21}a_{12} = \det(A).$$

For 2×2 matrices, the only assertions in Theorem 3.11 that have not been proved by the above are

(d) if B is obtained from A by multiplying row 2 by α then $\det(B) = \alpha \det(A)$;

(e) if B is obtained from A by adding a multiple of row 1 to row 2, then $\det(B) = \det(A)$.

Now, (d) follows by observing that B can also be obtained from A by first swapping row 1 and row 2, then multiplying row 1 by α and then swapping row 1 and row 2 again. Thus, by (a) and (b) above, $\det(B) = (-1) \cdot \alpha \cdot (-1) \det(A) = \alpha \det(A)$.

Finally, (e) follows by observing that B can also be obtained from A by first swapping row 1 and row 2, then adding a multiple row 2 to row 1 and then swapping row 1 and row 2 again. Thus, by (a) and (c) above, $\det(B) = (-1)^2 \det(A) = \det(A)$.

Exercise* 2.

(a) Since A is invertible

$$\det(A) \det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1,$$

$$\text{so } \det(A^{-1}) = (\det(A))^{-1}.$$

(b) By (a), we know that $\det(S^{-1}) = (\det(S))^{-1}$, thus

$$\det(B) = \det(S^{-1}AS) = \det(S^{-1}) \det(A) \det(S) = \frac{\det(S)}{\det(S)} \det(A) = \det(A).$$

Exercise* 3. Write the system in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ -8 \\ -3 \end{pmatrix}.$$

Now

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} \stackrel{R_3 - R_1}{=} \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 0 & 2 \end{vmatrix} = (-1) \cdot \begin{vmatrix} -3 & 1 \\ -2 & 2 \end{vmatrix} = -(-6 + 2) = 4,$$

$$\det(A_1(\mathbf{b})) = \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} \stackrel{R_3 - R_1}{=} \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -10 & 0 & 2 \end{vmatrix} = (-1) \cdot \begin{vmatrix} -8 & 1 \\ -10 & 2 \end{vmatrix} = -(-16 + 10) = 6,$$

$$\det(A_2(\mathbf{b})) = \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} \stackrel{R_3 - 2R_2}{=} \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 6 & 13 & 0 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 2 & 7 \\ 6 & 13 \end{vmatrix} = -(26 - 42) = 16,$$

$$\det(A_3(\mathbf{b})) = \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{R_3 - R_1}{=} \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ -2 & 0 & -10 \end{vmatrix} = (-1) \cdot \begin{vmatrix} -3 & -8 \\ -2 & -10 \end{vmatrix} = -(30 - 16) = -14.$$

Thus, by Cramer's rule,

$$x_1 = \frac{\det(A_1(\mathbf{b}))}{\det(A)} = \frac{6}{4} = \frac{3}{2}, \quad x_2 = \frac{\det(A_2(\mathbf{b}))}{\det(A)} = \frac{16}{4} = 4, \quad x_3 = \frac{\det(A_3(\mathbf{b}))}{\det(A)} = \frac{-14}{4} = -\frac{7}{2},$$

and the solution set of the system is $\{(\frac{3}{2}, 4, -\frac{7}{2})\}$.

Exercise* 4.

(a) We have

$$\det(A) = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & 2 \end{vmatrix} \stackrel{R_1 - 2R_2}{=} \begin{vmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \\ 0 & 5 & 1 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} = -(3 - 5) = 2 \neq 0,$$

so A is invertible.

(b) The $(3, 1)$ -cofactor of A is

$$C_{31} = (-1)^{3+1} \det(A_{31}) = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4,$$

so the $(1, 3)$ -entry of A^{-1} is

$$\frac{C_{31}}{\det(A)} = \frac{4}{2} = 2.$$

(c) First we need to calculate the remaining cofactors:

$$\begin{aligned} C_{11} &= + \begin{vmatrix} -1 & 1 \\ 4 & 2 \end{vmatrix} = -6, & C_{12} &= - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1, & C_{13} &= + \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5, \\ C_{21} &= - \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 10, & C_{22} &= + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1, & C_{23} &= - \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7, \\ C_{31} &= + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, & C_{32} &= - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, & C_{33} &= + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3. \end{aligned}$$

Thus

$$\text{adj}(A) = \begin{pmatrix} -6 & 10 & 4 \\ -1 & 1 & 1 \\ 5 & -7 & -3 \end{pmatrix}, \quad \text{and} \quad A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{pmatrix} -3 & 5 & 2 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{7}{2} & -\frac{3}{2} \end{pmatrix}.$$

Exercise* 5. In $C^1[a, b]$, axiom (C1) holds since the sum of two continuously differentiable functions on $[a, b]$ is again continuously differentiable on $[a, b]$, and axiom (C2) holds since a constant times a continuously differentiable function on $[a, b]$ is again continuously differentiable on $[a, b]$. The zero element in $C^1[a, b]$ is the zero function $\mathbf{0}(t) = 0$ for all $t \in [a, b]$. Given $\mathbf{f} \in C^1[a, b]$, then $-\mathbf{f}$ is the function $(-1)\mathbf{f}$, that is $(-\mathbf{f})(t) = -\mathbf{f}(t)$ for all $t \in [a, b]$, since then for all $t \in [a, b]$

$$(\mathbf{f} + (-\mathbf{f}))(t) = \mathbf{f}(t) + (-\mathbf{f})(t) = \mathbf{f}(t) - \mathbf{f}(t) = 0 = \mathbf{0}(t),$$

so $\mathbf{f} + (-\mathbf{f}) = \mathbf{0}$, that is, axiom (A4) holds.

Exercise 6. S is not a vector space with the indicated operations, since, for example, (A6) does not hold: if $\mathbf{u} = (1, 1) \in S$, then

$$(1 + 1)\mathbf{u} = 2\mathbf{u} = (2, 2) \quad \text{but} \quad \mathbf{u} \oplus \mathbf{u} = (2, 0).$$

In fact, (A3) cannot hold either (why?) and consequently (A4) does not even make sense.

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